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On the viscosity and thermal conduction of fluids with multivalued internal energy

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Abstract – This work is concerned with an extension of the classical compressible Euler model of fluid dynamics in which the fluid internal energy is a measure-valued quantity. This model can be derived from the hydrodynamic limit of a kinetic model involving a specific class of collision operators. In the present paper, we investigate diffusive corrections of this fluid dynamical model derived from a Chapman–Enskog expansion of the kinetic model, in the case where the collision time depends on the particle energy in the fluid frame. We show that the closure relations for the stress tensor and heat flux vector differ from their expression in the usual Navier–Stokes model. We argue why such a feature could be used as a tool towards an understanding of fluid turbulence from kinetic theory. © 2001 Éditions scientifiques et médicales Elsevier SAS

Boltzmann equation / Euler equation / Navier–Stokes equation / viscosity / heat conduction / Chapman–Enskog expansion / turbulence

1. Introduction

This work is concerned with a fluid-dynamical model which extends the classical Euler equations of compressible gas dynamics. It consists of a coupled system for the fluid mean velocity $u(x,t)$ on the one hand, and for the particle energy distribution function $g(x,\xi,t)$ on the other hand, where x and t are position and time and where $\xi = |v-u(x,t)|^2/2 \in [0,\infty)$ is the kinetic energy in the fluid rest frame of a particle of velocity v. This system of equations is written in dimension d ($d = 1, 2, 3$):

$$\frac{\partial g}{\partial t} + u\cdot\nabla_x g - \frac{2}{d}\,\xi\,\frac{\partial g}{\partial \xi}\,(\nabla_x\cdot u) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u) + \nabla_x(\rho u u) + \nabla_x\left(\frac{2W}{d}\right) = 0, \tag{2}$$

where ρ and W are the fluid number and energy densities, related to g through

$$\rho = \int_0^\infty g\,\mathrm{d}\nu(\xi), \qquad W = \int_0^\infty g\xi\,\mathrm{d}\nu(\xi), \tag{3}$$

with $\mathrm{d}\nu(\xi) = |\mathbb{S}^{d-1}||2\xi|^{(d-2)/2}\mathrm{d}\xi$ and $|\mathbb{S}^{d-1}|$ is the measure of the sphere in $\mathbb{R}^d$. The particle mass is set to 1 for simplicity. By integrating (1) with respect to ξ, a closed system of equations for the number and energy densities ρ and W and the mean velocity u is obtained, which turns out to be identical to the usual compressible Euler equations [1,2].

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It has been shown in [1,2] that this model can be formally derived from a hydrodynamic limit of a kinetic equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f = \frac{1}{\varepsilon} Q(f), \tag{4}$$

where $f = f(x, v, t)$ is the kinetic velocity distribution function and ε is the (supposedly small) Knudsen number. The collision operator $Q(f)$ describes the isotropization of the particle distribution function about the fluid mean velocity and is written

$$Q(f) = \frac{1}{\tau}\left(P_{u_f}(f) - f\right), \tag{5}$$

with

$$P_{u_f}(f) = \frac{1}{4\pi}\int_{S^2} f\left(u_f + |v - u_f|\omega\right) \mathrm{d}\omega, \quad u_f = \int vf \,\mathrm{d}v \left(\int f \,\mathrm{d}v\right)^{-1}, \tag{6}$$

where u_f is the average velocity of f and $\tau = \tau(x, t)$ is the mean collision time. The collision model (6) appears in space plasma physics as a simplified description of wave-particle interactions and is used in cosmic ray modeling [3–6]. The existence of solutions for (4) was proved in [7].

In the present paper we shall argue that, in a very preliminary and primitive way, this kinetic model could also provide a methodology for deriving compressible turbulence models from kinetic theory. Indeed, interpreting $f(x, v, t)$ as the velocity distribution of the small scale structures (rather than as a particle kinetic distribution) the operator (4) provides a model for the interactions of these structures provided that the energy dissipation occurs on a longer time scale than velocity isotropization. Since the interactions must be total momentum preserving, isotropization must naturally occur about the average fluid velocity.

In section 2, we shall show that indeed, system (1), (2) can be rephrased as a gas dynamics model in which the fluid internal energy $e = W/\rho$ is a measure-valued quantity. Therefore, the model (1), (2) retains the fact that within one mesoscopic scale $\mathrm{d}x$, a large number of fluid microstructures coexist, with an average velocity identical with the fluid mean velocity but with a full spectrum of allowed internal energies. In the laminar regime case, where no microstructure exists, the measure $\nu_{(x,t)}\,\mathrm{d}\xi$ reduces to a Dirac delta measure $\delta(\xi - e(x, t))$, where $e(x, t)$ is the usual (monovalued) fluid internal energy satisfying the standard compressible Euler equations.

We can expect that the diffusive effects (such as viscosity and heat conductivity) are modified by the multivaluedness of the internal energy. This can be a way to establish how turbulence (here represented by the 'continuous' spectrum of fluid internal energies) affects diffusion. In this direction, diffusive corrections to (1), (2) are derived in [2] by means of a Chapman–Enskog expansion of (4). As expected, this procedure does not lead to the compressible Navier–Stokes equations in their usual form: the heat flow vector appears to be a function of the higher-order ξ-moments of g. This indicates that the model is able to capture certain features of the microscopic structures in the expressions of the dissipative fluxes, in a similar fashion as other phenomenological turbulence models like the $K-\varepsilon$ model [8] do.

However, in [2], the viscosity appears similar as in the conventional Navier–Stokes model. This together with the fact that the isotropization time τ is independent of the energy of the microstructures indicates that the model (5)–(6) must be complexified in order to approach reality.

The major goal of the present paper is to investigate the influence of an energy-dependent isotropization time $\tau(\xi)$. This point is developped in sections 3 and 4. In this case, conservation of total momentum requires that isotropization occurs about an 'isotropization velocity' which may be different from the fluid average velocity. Near equilibrium ($\varepsilon \ll 1$), these two velocities are close up to the order ε. For this model, the viscosity as well as the heat flux appear dependent on the full spectrum of fluid energies represented by g. Therefore, a

ξ-dependent collision time τ leads to expressions of the diffusivities that seem to take better account of the presence of fluid microstructures than the constant τ model.

To be useful in practical cases, the model must not be too complex. The energy-distribution function g satisfies a second-order partial differential equation in a four-dimensional space (x, ξ), the numerical resolution of which would often be prohibitive. Therefore, in section 5, we shall propose closure strategies for the system of ξ-moments of g, which result in a lower-dimensional system of equations to solve.

Finally, we show in section 6 that our method, although seemingly restricted to relaxation time operators, is indeed more general and is valid for any rotationally invariant operator which preserves total momentum and the energy in an appropriate frame. In particular, our results also apply to Fokker–Planck type collision operators.

We end this introductory section by saying that the same methodology could be developped for incompressible fluids. We refer the reader to [2] for the case of a constant τ. The case of a ξ-dependent τ will be dealt with in future work. We also refer to [2] for a more detailed bibliography about the mathematical aspects of hydrodynamic and diffusion limits.

2. The fluid-dynamical model with measure-valued internal energy

In this section, we develop some considerations about the model (1), (2). We first show that it can be interpreted as a gas dynamics model with measure-valued internal energy. Let us consider the motion of a fluid element $\mathrm{d}x\,\mathrm{d}\xi$ in (x, ξ) space, according to equation (1), assuming that the velocity field $u(x, t)$ is given and smooth. We note that, throughout this paper, we shall disregard shock situations, where u is discontinuous. Shock theory for the multivalued energy model (1), (2) is an essentially open problem. With a smooth u, the trajectory $(X(t), \Xi(t))$ of this fluid element is a characteristic of (1) and consequently solves the following differential system:

$$\dot{X}(t) = u\big(X(t), t\big), \qquad \dot{\Xi}(t) = -\frac{2}{d}\Xi(t)(\nabla_x \cdot u)\big(X(t), t\big), \tag{7}$$

where the dot denotes the time derivative. We notice that the motion in x-space is independent of the position in the ξ-space and coincides with the convection of a usual fluid element by the velocity field u.

Let us denote by $(X(t; x, s), \Xi(t; x, \xi, s))$ the position at time t of the fluid element issued from (x, ξ) at time s. For a given pair (s, t), the mapping $x \to X(t; x, s)$ is a diffeomorphism of the position space. Its Jacobian $J(t; x, s)$ is the ratio of the volume elements: $\mathrm{d}X = J\,\mathrm{d}x$. According to a well-known result, J satisfies the differential equation

$$\dot{J}(t) = J(t)(\nabla_x \cdot u)\big(X(t), t\big), \tag{8}$$

But, using the second equation of (7), we can write

$$J^{-1}\dot{J} = (\nabla_x \cdot u)\big(X(t), t\big) = -\frac{d}{2}\Xi^{-1}\dot{\Xi},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\Xi^{d/2}J\big) = 0, \tag{9}$$

which means that $\Xi^{d/2}\,\mathrm{d}X$ is a constant of motion. Mathematically, one should say that the differential form $\omega(x, \xi) = \xi^{d/2}\,\mathrm{d}x$ is conserved by the flow in (x, ξ) space (i.e. at any time, its reciprocal image under the flow is equal to itself).

This identity reminds of another identity valid at the level of the usual compressible Euler equations. Indeed, the internal energy $e(x,t)$ is the solution of

$$\frac{\partial e}{\partial t} + (u\cdot\nabla_x)e - \frac{2}{d}e(\nabla_x\cdot u) = 0,$$

which implies that $E(t) = e(X(t),t)$ satisfies (9) with Ξ replaced by E and consequently that $E^{d/2}\,\mathrm{d}X$ is constant along a solution of the Euler equations (mathematically, one should say that the differential form $\alpha_t = e(x,t)\,\mathrm{d}x$ is convected by the flow, i.e. the reciprocal image of α_t by the flow is equal to α_0).

Both identities reflect the well-known evolution of the internal energy under an adiabatic compression or rarefaction. In other words, each fluid element $\mathrm{d}x\,\mathrm{d}\xi$ in (x,ξ) space evolves according to the same adiabatic law as the single fluid element $\mathrm{d}x$ in conventional gas dynamics. Therefore, one can view model (1) as the natural extension of usual gas dynamics when the fluid internal energy is multivalued. The picture can be summarized as follows: an arbitrary fluid volume element $\mathrm{d}x$ consists of infinitely many subelements, each of them having its own internal energy but sharing the same drift velocity as the 'big' fluid element. The probabilistic repartition of the internal energy ξ of the fluid subelements of the volume $\mathrm{d}x$ is obtained from g by constructing the probability measure

$$\nu_{(x,t)}(\mathrm{d}\xi) = \rho^{-1}(x,t)g(x,\xi,t)\,\mathrm{d}v(\xi), \qquad \mathrm{d}v(\xi) = \left|\mathbb{S}^{d-1}\right||2\xi|^{(d-2)/2}\,\mathrm{d}\xi. \tag{10}$$

The measure $\nu_{(x,t)}(\mathrm{d}\xi)$ expresses the probability that in a given fluid element $\mathrm{d}x$ at time t, one finds a subelement with internal energy in the range $(\xi,\xi+\mathrm{d}\xi)$. We obviously have

$$\int \nu_{(x,t)}(\mathrm{d}\xi) = 1, \qquad \int \xi\,\nu_{(x,t)}(\mathrm{d}\xi) = \frac{W}{\rho} = e.$$

Therefore, the fluid internal energy is just the average of the internal energy of the subelements with respect to the probability ν.

To be complete, this description must lead to the usual compressible Euler equations when the energy distribution of the fluid subelements is monovalued, i.e. when $\nu_{(x,t)}(\mathrm{d}\xi) = \delta(\xi - e(x,t))\,\mathrm{d}\xi$, or equivalently when

$$G := \left|\mathbb{S}^{d-1}\right||2\xi|^{(d-2)/2}g = \rho\delta\big(\xi - e(x,t)\big). \tag{11}$$

It is easy to see that G satisfies the conservative form of (1):

$$\frac{\partial G}{\partial t} + \nabla_x\cdot(uG) - \frac{\partial}{\partial\xi}\left(\frac{2}{d}\xi(\nabla_x\cdot u)G\right) = 0. \tag{12}$$

Now, it is an exercise to show that G satisfies (12), coupled with (2) if and only if the triple (n,u,e) satisfies the usual compressible Euler equation.

There is another particular solution of system (1), (2) which yields the gas dynamics equations: the Maxwellian

$$M_{\rho,T}(\xi) = \frac{\rho}{(2\pi T)^{d/2}}\exp(-\xi/T), \tag{13}$$

where T is the temperature. Again, it is an exercise to show that $M_{\rho,T}$ is a solution of (1) coupled with (2) if and only if (ρ,u,T) satisfies the usual compressible Euler equations.

It is interesting to remark that these two particular classes of solutions (11) and (13) do not overlap, and that in particular, (11) does not correspond to a zero temperature limit of (13). The delta function solutions (11) correspond to a phase-space distribution function $f(x, v, t)$ which is a measure supported by a sphere in velocity space, while the Maxwellian corresponds to the usual concept of thermodynamical equilibrium. While the Maxwellian naturally appears in the microscopic description of molecular motion, the delta solution is the lead from the multivalued energy model to the conventional one. This remark supports the use of the multivalued energy model as a convenient statistical model for turbulence modeling, as the statistical distribution is not related to the equilibrium statistics of the microscopic motion.

One of the major consequences of turbulence in a fluid flow is the increase of the diffusivities (viscosity and thermal conduction), due to the large gradients associated with the small scale turbulent motions. Since in many practical applications it is too expensive to resolve the small scales, various phenomenological models have been proposed. They attempt to describe the effects of the small scale unresolved turbulent motion on the large scale motion, and particularly on the diffusivities. They are based on a statistical treatment of the fluctuations of the fluid quantities and involve necessary closure assumptions (see, e.g., the $K - \varepsilon$ model [8]). We refer to [9–11] for general expositions of this problem.

Therefore, it is a natural question to investigate what are the shapes of the diffusivities associated with the model with multivalued energy (1), (2). Indeed, if the collision operator (5) is believed to provide a reasonable description of the interactions between the small scale structures, the diffusivities resulting from these interactions will be found by a higher order (Chapman–Enskog) expansion in ε of the kinetic equation (4). This has been performed in [2] for a constant collision time τ. In [2], it is shown that the multivalued energy model does not reduce to the conventional compressible Navier–Stokes equations by a simple moment procedure. Indeed, the resulting heat-flux vector is given in terms of a higher order moment of the energy distribution function, showing that heat dissipation depends on the full statistics of the fluid subelements. At variance, the viscous stress tensor is not modified from its Navier–Stokes value. This apparent insensitivity of the viscous stresses to the energy statistics of the fluid may be an artefact due to the crude simplicity of the collision model with a constant collision time.

The goal of the present paper is to prove that a more complex collision law, with a collision time τ depending on the relative energy in the fluid frame, actually yields diffusivities which are both (i.e. viscosity as well as heat dissipation) dependent on the energy statistics of the fluid. We first develop some considerations about the collision model in the next section.

3. The collision operator and the kinetic equation

3.1. The collision operator

We consider the following collision operator:

$$Q(f, u) = \frac{1}{\tau} L_u f, \tag{14}$$

with

$$L_u f(v) = \Pi_u(f)(v) - f(v) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f\big(u + |v - u|\omega'\big)\, \mathrm{d}\omega' - f(v). \tag{15}$$

The function $f = f(t, x, v)$ is the particle distribution function, depending on the position vector $x \in \mathbb{R}^d$, the velocity $v \in \mathbb{R}^d$, and the time $t > 0$. The vector $u = u_f(t, x)$ is some kind of average velocity of f which will

be specified below. The quantity $|v-u|^2/2$ is therefore the particle kinetic energy in a reference frame moving with velocity u and will be referred to as the 'relative kinetic energy'). Π_u is the orthogonal projector (in L^2) onto the space of functions depending on $|v-u|$ (or equivalently, on the relative kinetic energy $|v-u|^2/2$) only. Therefore $\Pi_u(f) = \Pi_u(f)(|v-u|^2/2)$. The collision time $\tau = \tau(t,x,|v-u|^2/2,[f])$ is supposed to depend locally upon x, t, and upon the relative kinetic energy $|v-u|^2/2$. Moreover, τ is an operator acting on f through its dependence with respect to v. Additionally, we assume that for any (t,x,v) and any f, $\tau(t,x,|v-u|^2/2,[f])$ only depends on v through $|v-u|^2/2$. An example of such τ is $\tau(P_u(f))$, but many other examples can be devised. Later on, we shall omit the dependences of τ upon x and t.

Now, we make the definition of u_f more precise. We determine u_f by the requirement that Q is momentum-preserving. To make it explicit, we first need the following:

PROPOSITION 3.1: (i) *We have*:

$$\int_{\mathbb{S}^{d-1}} Q(f,u)\big(u+|v-u|\omega\big)\,\mathrm{d}\omega = 0, \tag{16}$$

or equivalently:

$$\int_{\mathbb{R}^d} Q(f,u)(v)\phi\left(\frac{|v-u|^2}{2}\right)\mathrm{d}v = 0, \tag{17}$$

for all functions $\phi(\xi)$, $\xi>0$. In particular, $Q(f,u)$ preserves the density and the average fluid energy in the frame moving with velocity u, which is expressed by:

$$\int_{\mathbb{R}^d} Q(f,u)(v)\,\mathrm{d}v = 0, \qquad \int_{\mathbb{R}^d} Q(f,u)(v)\frac{|v-u|^2}{2}\,\mathrm{d}v = 0. \tag{18}$$

(ii) *The null-space of Q is the space of functions $\phi(\frac{|v-u|^2}{2})$ where the function $\phi(\xi)$ and the vector $u\in\mathbb{R}^d$ are arbitrary.*

(iii) *The collision operator* (14) *preserves momentum, i.e. satisfies*

$$\int_{\mathbb{R}^d} Q(f,u)(v)v\,\mathrm{d}v = 0, \tag{19}$$

if and only if the velocity u satisfies:

$$\int_{\mathbb{R}^d} (v-u)\,\tau^{-1}\left(\frac{|v-u|^2}{2},[f]\right)f(v)\,\mathrm{d}v = 0. \tag{20}$$

Then, the operator preserves the total fluid energy:

$$\int_{\mathbb{R}^d} Q(f,u)(v)\frac{|v|^2}{2}\,\mathrm{d}v = 0. \tag{21}$$

Proof. – (i) Since $\Pi_u(f)$ is a function of $|v-u|$ and since τ depends on v only through $|v-u|$, we have:

$$\frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} Q(f,u)\big(u+|v-u|\omega\big)\,\mathrm{d}\omega = \frac{1}{\tau}(\Pi_u f - \Pi_u f) = 0,$$

which gives equation (16). Now, we have

$$\int_{\mathbb{R}^d} Q(f,u)(v)\,\phi\big(|v-u|\big)\,\mathrm{d}v$$
$$= \int_0^{+\infty} \big|\mathbb{S}^{d-1}\big|\,\Pi_u\big(Q_u(f,u)\big)\,f\big(|v-u|\big)|v-u|^{d-1}\,\phi\big(|v-u|\big)d(|v-u|) = 0.$$

Relation (18) is obtained by taking $\phi(\xi)=1$ and $\phi(\xi)=\xi$ in (17).

(ii) If $f = f(|v-u|)$ then $\Pi_u f = f$ and $Q(f,u)=0$. Conversely, if $Q(f,u)=0$ then $f=\Pi_u f$ depends on v only through $|v-u|$.

(iii) Because the operator preserves the mass (18), we have

$$\int_{\mathbb{R}^d} v\,Q(f,u)(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} (v-u)\,Q(f,u)\,\mathrm{d}v = -\int_{\mathbb{R}^d} (v-u)\frac{1}{\tau} f(v)\,\mathrm{d}v.$$

We then deduce (20). The conservation of the total energy (21) comes from combining (18) and (19), using the identity $|v|^2/2 = |v-u|^2/2 - |u|^2/2 + v\cdot u$. □

In the remainder of the paper, we shall take $u = u_f$ satisfying (20) in the collision operator (14), (15).

Note that equation (20) defines u_f implicitly (because of the dependence of τ^{-1} upon u_f). The velocity u_f does not coincide with the usual fluid mean velocity $\bar{u} = \bar{u}_f$ defined by

$$\bar{u} = \frac{1}{\rho}\int_{\mathbb{R}^d} v f(v)\,\mathrm{d}v, \quad \rho = \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v. \tag{22}$$

In the particular case of a distribution function of the form $f = f(|v-\bar{u}|)$, the two concepts of average velocity coincide: $u_f = \bar{u}_f$ (more precisely, $\bar{u}_f$ is a solution of (20)). In this paper, we shall not dwell on the problem of solving (20) and we shall assume that there exists a unique 'physically admissible' velocity field solving (20).

3.2. The kinetic equation: approximate solutions and change to the local frame

We now consider the kinetic equation (4), with the collision operator defined by (14):

$$Tf^\varepsilon = \frac{1}{\varepsilon} Q\big(f^\varepsilon, u_{f^\varepsilon}\big), \qquad Tf \equiv \frac{\partial f}{\partial t} + v.\nabla_x f. \tag{23}$$

We define an approximate solution of (23) at the order n to be a solution $\tilde{f}^\varepsilon$ of

$$T\tilde{f}^\varepsilon = \frac{1}{\varepsilon} Q\big(\tilde{f}^\varepsilon, u_{\tilde{f}^\varepsilon}\big) + \mathrm{O}\big(\varepsilon^n\big). \tag{24}$$

We show that equation (20) for the definition of u_f can be equivalently replaced by an equation involving the first moments of f.

Indeed, equation (23) is equivalent to the following system, of unknowns f^ε and u^ε:

$$\begin{cases} Tf^\varepsilon = \dfrac{1}{\varepsilon} Q\big(f^\varepsilon, u^\varepsilon\big), \\ \displaystyle\int_{\mathbb{R}^d} \big(v-u^\varepsilon\big)\tau^{-1}\left(\frac{|v-u^\varepsilon|^2}{2}, f^\varepsilon\right) f^\varepsilon(v)\,\mathrm{d}v = 0. \end{cases} \tag{25}$$

Since the second equation of (25) is equivalent to to the fact that Q is momentum-conservative, system (25) is equivalent to:

$$\begin{cases} Tf^\varepsilon = \dfrac{1}{\varepsilon} Q(f^\varepsilon, u^\varepsilon), \\ \displaystyle\int_{\mathbb{R}^d} v T f^\varepsilon(v)\,\mathrm{d}v = 0. \end{cases} \tag{26}$$

Then, u^ε appears as the Lagrange multiplier of the momentum preservation constraint as expressed by the second equation of (26).

The same splitting can be performed for the order n approximate solutions: indeed, $\tilde{f}^\varepsilon$ is an order n approximate solution according to definition (24) if and only if there exists $\tilde{u}^\varepsilon$ such that $(\tilde{f}^\varepsilon, \tilde{u}^\varepsilon)$ satisfies:

$$\begin{cases} T\tilde{f}^\varepsilon = \dfrac{1}{\varepsilon} Q(\tilde{f}^\varepsilon, \tilde{u}^\varepsilon) + \mathrm{O}(\varepsilon^n), \\ \displaystyle\int_{\mathbb{R}^d} v T\tilde{f}^\varepsilon(v)\,\mathrm{d}v = \mathrm{O}(\varepsilon^n). \end{cases} \tag{27}$$

The 'only if' part is obvious with the choice $\tilde{u}^\varepsilon = u_{\tilde{f}^\varepsilon}$ by multiplying (24) by v, integrating it with respect to v and using that

$$\int_{\mathbb{R}^d} v Q(\tilde{f}^\varepsilon, u_{\tilde{f}^\varepsilon}) = 0. \tag{28}$$

Conversely, from (27) and (28), we have

$$\int_{\mathbb{R}^d} v Q(\tilde{f}^\varepsilon, \tilde{u}^\varepsilon) - \int_{\mathbb{R}^d} v Q(\tilde{f}^\varepsilon, u_{\tilde{f}^\varepsilon}) = \mathrm{O}(\varepsilon^{n+1}).$$

Thus, under the hypothesis that equation (20) has a unique branch of 'physically admissible' solutions and that this branch consists of regular solutions of the non-linear equation (20), we formally get:

$$\tilde{u}^\varepsilon = u_{\tilde{f}^\varepsilon} + \mathrm{O}(\varepsilon^{n+1}).$$

This shows that every solution $\tilde{f}^\varepsilon$ of (27) is a solution to (24).

We now transform the approximate solutions defined by (27) by evaluating the kinetic velocities in the frame moving with velocity $\tilde{u}^\varepsilon$. We introduce:

$$p = v - \tilde{u}^\varepsilon, \qquad \tilde{F}^\varepsilon(p) = \tilde{f}^\varepsilon(v), \qquad \xi = \frac{|v - \tilde{u}^\varepsilon|^2}{2}. \tag{29}$$

To simplify the notations, we shall omit the subscript ε, the tildes and the arguments t and x of the function τ whenever the context is clear. We still denote by f and u the solutions to (27), and by F the corresponding function obtained by the change of variable (29). In terms of $F = F(p)$, we have

$$Tf = \frac{\partial F}{\partial t} + u \cdot \nabla_x F + p \cdot \nabla_x F - \left(\frac{\partial u}{\partial t} + (\nabla_x u) u \right) \cdot \nabla_p F - (\nabla_x u)\, p \cdot \nabla_p F,$$

where $\nabla_x u$ denotes the matrix whose coefficients are $(\nabla_x u)_{ij} = \partial u_i/\partial x_j$. Then the first equation of (27) becomes:

$$\mathcal{A}(F,u) = \frac{1}{\varepsilon}\frac{1}{\tau(\xi,F)}LF + \mathrm{O}(\varepsilon^n), \tag{30}$$

where L is the operator L_0 given by (15) for $u = 0$:

$$LF(p) = \Pi(F)(p) - F(p) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} F(|p|\omega')\,\mathrm{d}\omega' - F(p), \tag{31}$$

and

$$\begin{cases} \mathcal{A}(F,u) = T_u F + p\cdot\nabla_x F - C_u\cdot\nabla_p F - (\nabla_x u)p\cdot\nabla_p F, \\ T_u F = \dfrac{\partial F}{\partial t} + u\cdot\nabla_x F, \\ C_u = \dfrac{\partial u}{\partial t} + (\nabla_x u)u. \end{cases} \tag{32}$$

Now we recall that u and f are linked by the second equation (27), which can be written in terms of F:

$$\int_{\mathbb{R}^d} p\mathcal{A}(F,u)(p)\,\mathrm{d}p = \mathrm{O}(\varepsilon^n). \tag{33}$$

Using the expression of $\mathcal{A}$, we obtain by a simple integration:

$$\int_{\mathbb{R}^d} p\mathcal{A}(F,u)(p)\,\mathrm{d}p = \mathcal{B}(F,u),$$

with

$$\mathcal{B}(F,u) = C_u\int_{\mathbb{R}^d} F\,\mathrm{d}p + \nabla_x\cdot\left(\int_{\mathbb{R}^d} p\otimes pF\,dp\right) + \left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)\left(\int_{\mathbb{R}^d} pF\,\mathrm{d}p\right). \tag{34}$$

Finally, problem (27) is equivalent to the following system of equations:

$$\begin{cases} \tau(\xi,F)\mathcal{A}(F,u) = \dfrac{1}{\varepsilon}LF + \mathrm{O}(\varepsilon^n), \\ \mathcal{B}(F,u) = \mathrm{O}(\varepsilon^n), \end{cases} \tag{35}$$

with $\mathcal{A}(F,u)$ given by (32) and $\mathcal{B}(F,u)$ by (34).

In (35), the collision operator L is linear and independent of u. L is clearly a self-adjoint operator (in L^2) and its null-space is simply the space of isotropic functions (i.e. the functions depending on p only through $|p|$). Note also that the implicit character of equation (20) is now concentrated in the second equation of (35): for a given F, the corresponding u is simply a solution of an equation whose coefficients are averages of the distribution function F with respect to p.

4. Approximate macroscopic models to the first and second orders

4.1. The Chapman–Enskog expansion, review and notations

In this subsection, we first review some basic facts about the Chapman–Enskog expansion. This method allows to find successive approximate solutions to kinetic equations of the general form (4) in terms of the small parameter ε. To simplify the general presentation, we shall restrict ourselves to a linear $Q(f)$ although the method also standardly applies to non-linear operators. The presentation below is inspired from [12]. We also refer to [13–16] for general expositions about fluid-dynamical limits of kinetic theory.

The null-space $N(Q^*)$ of the adjoint Q^* of the collision operator plays a central role. We denote by Π the L^2-orthogonal projection onto $N(Q^*)$. The choice of the notation Π is not coincidental because in our example (14), Π is actually given by (15). We also denote by $\langle f, g\rangle$ the usual scalar product between two elements f and g of L^2.

The Chapman–Enskog method consists in looking for an order n approximate solution in the sense (24) of the following form:

$$f^{\varepsilon,n} = f_0^\varepsilon + \varepsilon f_1^\varepsilon + \varepsilon^2 f_2^\varepsilon + \cdots + \varepsilon^n f_n^\varepsilon, \tag{36}$$

where the functions f_k^ε may depend on ε but remain $\mathrm{O}(1)$ as $\varepsilon \to 0$. Below we will see that f_0^ε is an equilibrium function ($f_0^\varepsilon \in N(Q)$), and that the f_k^ε, $k \geqslant 1$, are successive corrections of f_0^ε and are taken in $N(Q)^\perp$. For simplicity we will omit the subscript ε in all the sequel.

For an approximation of order 1, we only have to choose (formally) f_0 and f_1 such that:

$$Q(f_0) = 0 \quad \text{and} \quad Tf_0 = Q(f_1).$$

For these equations to admit solutions we must choose f_0 in $N(Q)$ such that $Tf_0 \in N(Q^*)^\perp$, which also reads:

$$f_0 \in N(Q) \quad \text{and} \quad \Pi T f_0 = 0. \tag{37}$$

The solvability equation (37) leads to the macroscopic model associated with (4) at the first order in ε.

At higher orders, a straightforward identification between terms of the same order in ε (i.e. the Hilbert expansion) does not lead to the expected expression of the diffusive fluxes (such as, e.g., viscosity and thermal conduction). To that aim, a more complex matching of the terms of the various orders must be performed. First we write

$$Tf_0 + \varepsilon Tf_1 + \varepsilon^2 Tf_2 = \frac{1}{\varepsilon}\big\{Q(f_0) + \varepsilon Q(f_1) + \varepsilon^2 Q(f_2)\big\} + \mathrm{O}\big(\varepsilon^2\big), \tag{38}$$

which implies in particular that

$$\Pi T f_0 = \mathrm{O}(\varepsilon). \tag{39}$$

Thus the quantity $\Pi T f_0$ can be substracted from the zeroth-order term and added to the first-order one. We then obtain by identification:

$$\begin{aligned} &Q(f_0) = 0, \\ &Tf_0 - \Pi T f_0 = Q(f_1), \\ &Tf_1 + \frac{1}{\varepsilon}\Pi T f_0 = Q(f_2). \end{aligned} \tag{40}$$

The second equation of (40) satisfies the solvability condition by construction. It can be solved by $f_1 = Q^{-1}(Tf_0 - \Pi Tf_0)$ with Q^{-1} the inverse of the restriction of Q to $N(Q)^\perp$. However, for the third equation to have a solution, one has to write a solvability condition:

$$\Pi Tf_0 + \varepsilon \Pi T f_1 = \Pi Tf_0 + \varepsilon \Pi T Q^{-1}(I - \Pi)Tf_0 = 0. \tag{41}$$

In the case of collisions operators of Boltzmann type, it is well-known that the equation (41) is equivalent to the system of Navier–Stokes equations with diffusion terms proportional to ε.

The Chapman–Enskog method can be pursued to higher orders, following the same process as for the order 2. The order $n \geqslant 3$ expansion leads to the following solvability condition on f_0:

$$\Pi Tf_0 + \varepsilon \Pi T Q^{-1}(I - \Pi)Tf_0 + \sum_{i=2}^{n-1} \varepsilon^i \Pi T \left[Q^{-1}(I - \Pi)T\right]^i f_0 = 0. \tag{42}$$

However, in the present paper, we shall restrict to first- and second-order approximate solutions because higher-order fluid models (like the third-order or 'Burnett' model) are generally ill-posed.

4.2. The multivalued energy model to first order

In this section, we check that system (1), (2) is indeed the first order approximate model of (23), and we use the formulation (35). According to the Chapman–Enskog method, we expand $F = F_0 + \varepsilon F_1$ and insert this expression into system (35):

$$\begin{cases} \tau(\xi, F_0 + \varepsilon F_1)\left[\mathcal{A}(F_0, u) + \varepsilon \mathcal{A}(F_1, u)\right] = \dfrac{1}{\varepsilon} L F_0 + L F_1 + \mathrm{O}(\varepsilon), \\ \mathcal{B}(F_0, u) + \varepsilon \mathcal{B}(F_1, u) = \mathrm{O}(\varepsilon). \end{cases} \tag{43}$$

Identifying terms of the same order in ε and removing the terms of order ε, we obtain

$$\begin{cases} L F_0 = 0, \\ \tau \mathcal{A}(F_0, u) = L F_1, \\ \mathcal{B}(F_0, u) = 0. \end{cases} \tag{44}$$

The first equation of (44) implies that F_0 is a function of $\xi = |p|^2/2$ only: $F_0(p) = g(\xi)$. The second equation of (44) admits a solution F_1 if and only if $\Pi \mathcal{A}(F_0, u) = 0$. We have

$$\mathcal{A}(F_0, u) = \mathcal{A}(g, u) = T_u g + p \cdot \left(\nabla_x g - C_u \frac{\partial g}{\partial \xi}\right) - \left[(\nabla_x u) p \cdot p\right] \frac{\partial g}{\partial \xi}, \tag{45}$$

with T_u and C_u given in (32). To make $\Pi \mathcal{A}(g, u)$ and $\mathcal{B}(g, u)$ explicit, we use the following elementary lemma (the proof of which is omitted):

LEMMA 4.1: *If ϕ is a function of $\xi = \frac{|p|^2}{2}$, then we have*:

$$\Pi\left[p \otimes p \phi(\xi)\right] = \frac{2\xi}{d} \phi(\xi) I, \qquad \Pi\left[\mathcal{M} p \cdot p \phi(\xi)\right] = \frac{2\xi}{d} \mathrm{Tr}(\mathcal{M}) \phi(\xi),$$

and

$$\Pi\big[(\mathcal{M}p\cdot p)p\otimes p\,\phi(\xi)\big]=\frac{4\xi^2}{d(d+2)}\big[\mathrm{Tr}(\mathcal{M})I+\mathcal{M}+\mathcal{M}^T\big]\phi(\xi),$$

$$\Pi\big[(\mathcal{M}p\cdot p)(\mathcal{N}p\cdot p)\phi(\xi)\big]=\frac{4\xi^2}{d(d+2)}\big[\mathrm{Tr}(\mathcal{M})\mathrm{Tr}(\mathcal{N})+(\mathcal{M}+\mathcal{M}^T):\mathcal{N}\big]\phi(\xi),$$

for any $d\times d$ matrices $\mathcal{M}$ and $\mathcal{N}$ possibly depending on (x,t). We have denoted by $A:B$ the contracted product between two $d\times d$ matrices A and B ($A:B=\mathrm{Tr}(AB)$), by $\mathrm{Tr}(A)$ the trace of the matrix A and by I the identity $d\times d$ matrix.

Then, we easily get:

$$\Pi\mathcal{A}(g,u)=T_u g-\frac{2}{d}(\nabla_x\cdot u)\xi\frac{\partial g}{\partial\xi},\qquad \mathcal{B}(g,u)=\rho C_u+\nabla_x\left(\frac{2W}{d}\right). \tag{46}$$

It follows that system (44) is equivalent to system (1), (2) for g and u. We recover the same result as in [1] and [2]. In particular, at this order of approximation, the discrepancy between the fluid mean velocity $\bar{u}_f$ and the isotropization velocity u_f is not detectable.

4.3. The multivalued energy fluid model to second order

We seek an order two approximate solution to (26). We recall that this corresponds to a solution (F,u) to system (35) for $n=2$. As for the first order, we assume that F has the form:

$$F=g(t,x,\xi)+\varepsilon F_1+\varepsilon^2F_2, \tag{47}$$

and insert this expression into equations (35)

$$\begin{cases}\tau(\xi,F)\{\mathcal{A}(g,u)+\varepsilon\mathcal{A}(F_1,u)+\varepsilon^2\mathcal{A}(F_2,u)\}=LF_1+\varepsilon LF_2+\mathrm{O}(\varepsilon^2),\\ \mathcal{B}(g,u)+\varepsilon\mathcal{B}(F_1,u)+\varepsilon^2\mathcal{B}(F_2,u)=\mathrm{O}(\varepsilon^2).\end{cases} \tag{48}$$

The first equation implies in particular that $\Pi\mathcal{A}(g,u)=\mathrm{O}(\varepsilon)$. Thus the first equation of (48) can be rewritten in the form

$$\tau(\xi,F)\left\{(I-\Pi)\mathcal{A}(g,u)+\varepsilon\left[\mathcal{A}(F_1,u)+\frac{1}{\varepsilon}\Pi\mathcal{A}(g,u)\right]+\varepsilon^2\mathcal{A}(F_2,u)\right\}=LF_1+\varepsilon LF_2+\mathrm{O}(\varepsilon^2). \tag{49}$$

We identify the terms of the same order in ε and remove the terms of order 2, and get

$$\begin{cases}\tau(\xi,F)(I-\Pi)\mathcal{A}(g,u)=LF_1,\\ \tau(\xi,F)\left(\mathcal{A}(F_1,u)+\dfrac{1}{\varepsilon}\Pi\mathcal{A}(g,u)\right)=LF_2,\\ \mathcal{B}(g,u)+\varepsilon\mathcal{B}(F_1,u)=0.\end{cases} \tag{50}$$

Now we remark that the restriction of L to $N(L)^\perp$ is simply $-I$ and then its inverse L^{-1} is also equal to $-I$. If we seek $F_1\in N(L)^\perp$, the solution of the first equation (50) is

$$F_1=-\tau(\xi,F)(I-\Pi)\mathcal{A}(g,u). \tag{51}$$

Then, the second equation of (50) has a non-empty set of solutions F_2 if and only if the projection Π of the left-hand side vanishes. Therefore, the last two equations of (50) are equivalent to

$$\begin{cases} \Pi\mathcal{A}(g,u) + \varepsilon\Pi\mathcal{A}(F_1,u) = 0, \\ \mathcal{B}(g,u) + \varepsilon\mathcal{B}(F_1,u) = 0. \end{cases} \tag{52}$$

Now, the expression of F_1 contains a dependence on $F = g + \varepsilon F_1 + \varepsilon^2 F_2$ through the function τ. But F_1 only appears in terms of order 1 in ε in (52). Therefore, with the same accuracy, we can replace $\tau(F)$ by $\tau(g)$ in (51) and let

$$F_1 = -\tau(\xi, g)(I - \Pi)\mathcal{A}(g,u). \tag{53}$$

The two equations (52) with F_1 given by (53) provide a sufficient condition on the pair (g,u) which makes F given by (47) an order 2 approximate solution of the kinetic model (23). Furthermore, any order 2 approximate solution of the form (47) with $F_1,\ F_2 \in N(L)^\perp$ is given by equation (52) as the next proposition shows:

PROPOSITION 4.2: *The pair $F = g + \varepsilon F_1 + \varepsilon^2 F_2$, u with $\Pi F_1 = \Pi F_2 = 0$, is an order 2 approximate solution to the kinetic equation* (35) *if and only if (g,u) is a solution of* (52)*.*

Proof. – We insert the expansion $F = g + \varepsilon F_1 + \varepsilon^2 F_2$ into (35) and obtain (48). For the first equation of (48), we take the orthogonal projection onto $N(L)$ and obtain:

$$\Pi\big[\mathcal{A}(g,u) + \varepsilon\mathcal{A}(F_1 + \varepsilon F_2, u)\big] = \mathrm{O}\big(\varepsilon^2\big). \tag{54}$$

Then the first equation of (48) is equivalent to:

$$L(F_1 + \varepsilon F_2) = \tau(\xi, F)\big[(I - \Pi)\mathcal{A}(g,u) + \varepsilon(I - \Pi)\mathcal{A}(F_1 + \varepsilon F_2, u)\big] = \mathrm{O}\big(\varepsilon^2\big). \tag{55}$$

Because $F_1,\ F_2 \in N(L)^\perp$, we can now apply L^{-1} and obtain:

$$F_1 + \varepsilon F_2 = \tau L^{-1}\big[(I - \Pi)\mathcal{A}(g,u) + \varepsilon(I - \Pi)\mathcal{A}(F_1 + \varepsilon F_2, u)\big] + \mathrm{O}\big(\varepsilon^2\big). \tag{56}$$

We insert the expression of $F_1 + \varepsilon F_2$ given by (56) into (54) and get

$$\Pi\big[\mathcal{A}(g,u) + \varepsilon\mathcal{A}\big(\tau L^{-1}(I - \Pi)\mathcal{A}(g,u), u\big)\big] = \mathrm{O}\big(\varepsilon^2\big).$$

With the second equation of (48), we finally obtain:

$$\begin{cases} \Pi\big[\mathcal{A}(g,u) + \varepsilon\mathcal{A}\big(\tau L^{-1}(I - \Pi)\mathcal{A}(g,u), u\big)\big] = \mathrm{O}\big(\varepsilon^2\big), \\ \mathcal{B}(g,u) + \varepsilon\mathcal{B}\big(\tau L^{-1}(I - \Pi)\mathcal{A}(g,u), u\big) = \mathrm{O}\big(\varepsilon^2\big). \end{cases} \tag{57}$$

We can now replace $\tau(F)$ by $\tau(g)$ because we are interested only in terms of order less than 1 in ε. This leads to (52). Therefore, such an order two approximate solution is necessarily given by (52). □

Equations (52) lead to the following model:

PROPOSITION 4.3: *System* (52) *is equivalent to the following system*:

$$\begin{cases} \dfrac{\partial g}{\partial t} + u\cdot\nabla_x g - (\nabla_x\cdot u)\dfrac{2\xi}{d}\dfrac{\partial g}{\partial \xi} = \varepsilon\dfrac{2}{d}\xi^{1-d/2}\left(\nabla_x - C_u\dfrac{\partial}{\partial\xi}\right)\left(\tau(\xi,g)\xi^{d/2}\left(\nabla_x - C_u\dfrac{\partial}{\partial\xi}\right)g\right) \\ \quad + \varepsilon\dfrac{4}{d(d+2)}\xi^{1-d/2}\dfrac{\partial}{\partial\xi}\left(\tau(\xi,g)\xi^{\frac{d+2}{2}}\dfrac{\partial g}{\partial\xi}\right)\big(\sigma(u):(\nabla_x u)\big), \\ \rho\left(\dfrac{\partial\bar u}{\partial t} + (\nabla_x\bar u)(\bar u)\right) + \dfrac{2}{d}\nabla_x W = \varepsilon\nabla_x\cdot\big(\mu_\tau\sigma(\bar u)\big), \end{cases} \tag{58}$$

where u and $\bar u$ are linked by:

$$\rho(\bar u - u) = -\varepsilon\int_{\mathbb{R}^d}\frac{2\xi}{d}\tau\left(\nabla_x g - C_u\frac{\partial g}{\partial\xi}\right)\mathrm{d}p \tag{59}$$

and with:

$$\begin{pmatrix}\rho(x,t)\\ W(x,t)\end{pmatrix} = \int_{\mathbb{R}^d} g(x,\xi,t)\begin{pmatrix}1\\ \xi\end{pmatrix}\mathrm{d}p, \quad \sigma(u) = \nabla_x u + (\nabla_x u)^T - \frac{2}{d}(\nabla_x\cdot u)I \tag{60}$$

(*where the exponent T denotes the transpose of a matrix*) *and*

$$\mu_\tau = \frac{-1}{d(d+2)}\int_{\mathbb{R}^d}|p|^4\tau(\xi,g)\frac{\partial g}{\partial\xi}\mathrm{d}p. \tag{61}$$

We recall that $\mathrm{d}p = |\mathbb{S}^{d-1}||2\xi|^{(d-2)/2}\mathrm{d}\xi$.

Before proving this proposition, we comment on the result. The second equation (58) is the fluid momentum conservation equation in a form close to the usual Navier–Stokes equation, except for the expression of the viscosity (61). The first equation (58) gives the diffusive corrections to the equation (1). There are two dissipative terms at the right-hand side, respectively corresponding to the heat dissipation and to the viscous force. The heat dissipation operator appears as a degenerate diffusion operator acting along oblique lines in the (x,ξ)-space, whose slopes are equal to the acceleration of a fluid element C_u. Note that, from the second equation (58), up to terms of order ε^2, it is possible to replace C_u in the first equation of (58) by $-2(\rho d)^{-1}\nabla_x W$. The viscous forces produce a diffusion in the ξ-space only.

The model (58) differs from that derived in [2] not only in the expression of the viscosity and in the fact that the ξ-dependent τ appears in the equation for g, but essentially in the discrepancy between the two velocities $\bar u$ and u. We shall see (see remark 1 below) that up to terms of order ε^2, $\bar u$ is the usual fluid mean velocity. From (59), $\bar u$ differs from the isotropization velocity u by a term of order ε, which involves the derivative of g along the same lines in (x,ξ) space as those appearing in the degenerate heat dissipation operator. Furthermore, $\bar u - u$ vanishes in the case of a constant τ because τ can be taken out of the ξ integral and what remains vanishes identically by an integration by parts. Therefore, the discrepancy appears as the result of an interplay between the ξ-dependent τ and the existence of gradients of g along certain prefered lines in the (x,ξ) space. Further mechanical interpretations of this model will be developed in future work.

Proof. – We just have to prove that system (52) is equivalent to system (58). We already know the expression of $\Pi\mathcal{A}(g,u)$ from (46). Using (53), we have:

$$F_1 = -\tau(I-\Pi)\mathcal{A}(g,u) = -\tau p\cdot\left(\nabla_x g - C_u\frac{\partial g}{\partial\xi}\right) + \tau[\mathcal{U}p\cdot p]\frac{\partial g}{\partial\xi}, \tag{62}$$

with $\mathcal{U}$ being the following traceless tensor

$$\mathcal{U}(t,x) = \nabla_x u - d^{-1}(\nabla_x \cdot u) I. \tag{63}$$

Using the expression (32) of the operator $\mathcal{A}$, we then obtain:

$$\begin{aligned}\mathcal{A}(F_1,u) = {} & T_u\left(\mathcal{U}p\cdot p\tau(g)\frac{\partial g}{\partial \xi}\right) - p\cdot\nabla_x\left(p.\tau(g)\nabla_x g\right)\\ & + p\cdot\nabla_x\left(C_u\cdot p\tau(g)\frac{\partial g}{\partial\xi}\right) + C_u\cdot\nabla_p\left(p\cdot\tau(g)\nabla_x g\right) - C_u\cdot\nabla_p\left(C_u\cdot p\tau(g)\frac{\partial g}{\partial\xi}\right)\\ & - (\nabla_x u)p\cdot\nabla_p\left(\mathcal{U}p.p\tau(g)\frac{\partial g}{\partial\xi}\right) + G(p),\end{aligned} \tag{64}$$

where $G(p)$ only involves odd terms with respect to p. In particular, we have $\Pi G = 0$. By developing the derivatives with respect to p, we get

$$\begin{aligned}\mathcal{A}(F_1,u) = {} & T_u\left(\mathcal{U}p\cdot p\tau(g)\frac{\partial g}{\partial\xi}\right) - \nabla_x\cdot\left((p\otimes p)\tau(g)\nabla_x g\right) + C_u\cdot\tau(g)\nabla_x g\\ & + \nabla_x\cdot\left(p\otimes pC_u\tau(g)\frac{\partial g}{\partial\xi}\right) + C_u\cdot\left(p\otimes p\frac{\partial}{\partial\xi}\left(\tau(g)\nabla_x g\right)\right) - |C_u|^2\tau(g)\frac{\partial g}{\partial\xi}\\ & - C_u\cdot p\otimes pC_u\frac{\partial}{\partial\xi}\left(\tau(g)\frac{\partial g}{\partial\xi}\right) - (\nabla_x u)p\cdot\left(\mathcal{U}+\mathcal{U}^T\right)p\tau(g)\frac{\partial g}{\partial\xi}\\ & - \left((\nabla_x u)p\cdot p\right)(\mathcal{U}p\cdot p)\frac{\partial}{\partial\xi}\left(\tau(g)\frac{\partial g}{\partial\xi}\right) + G(p).\end{aligned} \tag{65}$$

Now, to take the projection of this expression, we need lemma 4.1. We get:

$$\begin{aligned}\Pi\mathcal{A}(F_1,u) = {} & -\nabla_x\cdot\left(\frac{2\xi}{d}\tau(g)\nabla_x g\right) + C_u\cdot\tau(g)\nabla_x g + \nabla_x\cdot\left(\frac{2\xi}{d}C_u\tau(g)\frac{\partial g}{\partial\xi}\right)\\ & + C_u\cdot\left(\frac{2\xi}{d}\frac{\partial}{\partial\xi}(\tau(g)\nabla_x g)\right) - |C_u|^2\left(\tau(g)\frac{\partial g}{\partial\xi} - \frac{2\xi}{d}\frac{\partial}{\partial\xi}\left(\tau(g)\frac{\partial g}{\partial\xi}\right)\right)\\ & - \left(\mathcal{U}+\mathcal{U}^T\right):(\nabla_x u)\frac{2\xi}{d}\tau(g)\frac{\partial g}{\partial\xi} - \left(\mathcal{U}+\mathcal{U}^T\right):(\nabla_x u)\frac{4\xi^2}{d(d+2)}\frac{\partial}{\partial\xi}\left(\tau(g)\frac{\partial g}{\partial\xi}\right).\end{aligned} \tag{66}$$

Introducing the traceless rate-of-strain tensor $\sigma(u) = \mathcal{U} + \mathcal{U}^T$ also given by (60), and factorizing the expression (66), we obtain:

$$\begin{aligned}\Pi\mathcal{A}(F_1,u) = {} & -\frac{2}{d}\xi^{1-d/2}\left(\nabla_x - C_u\frac{\partial}{\partial\xi}\right)\left(\tau(\xi,g)\xi^{d/2}\left(\nabla_x - C_u\frac{\partial}{\partial\xi}\right)g\right)\\ & - \frac{4}{d(d+2)}\xi^{1-d/2}\frac{\partial}{\partial\xi}\left(\tau(\xi,g)\xi^{\frac{d+2}{2}}\frac{\partial g}{\partial\xi}\right)\left(\sigma(u):(\nabla_x u)\right).\end{aligned} \tag{67}$$

Inserting the above found expressions for $\Pi\mathcal{A}(g,u)$ and $\Pi\mathcal{A}(F_1,u)$ into the first equation (52) leads to the first equation (58).

Now, to make the second equation of system (52) explicit, we need to compute $\mathcal{B}(g,u)$ and $\mathcal{B}(F_1,u)$. $\mathcal{B}(g,u)$ is given by (46). With the expression of F_1 given by (62), we also have

$$\begin{aligned}\mathcal{B}(F_1,u) \;=\; & \nabla_x\cdot\Big(\int_{\mathbb{R}^d}(\mathcal{U}p\cdot p)p\otimes p\tau(g)\frac{\partial g}{\partial\xi}\,\mathrm{d}p\Big)\\ & -\Big(\frac{\partial}{\partial t}+u\cdot\nabla_x+\nabla_x u+(\nabla_x\cdot u)I\Big)\Big(\int_{\mathbb{R}^d}p\otimes p\Big(\tau(g)\nabla_x g-C_u\tau\frac{\partial g}{\partial\xi}\Big)\,\mathrm{d}p\Big),\end{aligned}\tag{68}$$

and then, using lemma 4.1:

$$\mathcal{B}(F_1,u)=-\nabla_x\cdot\big(\mu_\tau\sigma(u)\big)-\Big(\frac{\partial}{\partial t}+u\cdot\nabla_x+\nabla_x u+(\nabla_x\cdot u)I\Big)\Big(\int_{\mathbb{R}^d}\frac{2\xi}{d}\Big(\tau(g)\nabla_x g-C_u\tau\frac{\partial g}{\partial\xi}\Big)\,\mathrm{d}p\Big),\tag{69}$$

with $\sigma(u)$ defined by (60) and μ_τ by (61). We deduce that the second equation (52) leads to

$$\begin{aligned}&\rho\Big(\frac{\partial u}{\partial t}+(\nabla_x u)(u)\Big)+\frac{2}{d}\nabla_x W\\ &\quad=\varepsilon\nabla_x\cdot\big(\mu_\tau\sigma(u)\big)+\varepsilon\Big(\frac{\partial}{\partial t}+u\cdot\nabla_x+\nabla_x u+(\nabla_x\cdot u)I\Big)\Big(\int_{\mathbb{R}^d}\frac{2\xi}{d}\tau(\xi,g)\Big(\nabla_x g-C_u\frac{\partial g}{\partial\xi}\Big)\,\mathrm{d}p\Big).\end{aligned}\tag{70}$$

Now, introducing $\bar u$ according to (59) and integrating the first equation of (70) with respect to $\mathrm{d}p=|\mathbb{S}^{d-1}|(2\xi)^{(d-2)/2}\,\mathrm{d}\xi$, we obtain:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho u)=\varepsilon\nabla_x\cdot\int_{\mathbb{R}^d}\frac{2\xi}{d}\tau\Big(\nabla_x g-C_u\frac{\partial g}{\partial\xi}\Big)\,\mathrm{d}p=-\nabla_x\cdot\big(\rho(\bar u-u)\big),\tag{71}$$

which is nothing else than the continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\bar u)=0.\tag{72}$$

Then, repeatedly using the continuity equation (72) in the following computations, we have:

$$\begin{aligned}\rho C_u \;+\; & \Big(\frac{\partial}{\partial t}+u\cdot\nabla_x+\nabla_x u+(\nabla_x\cdot u)I\Big)(\rho\bar u-\rho u)\\ &=\big[\nabla_x\cdot(\rho\bar u)\big]u+\Big(\frac{\partial}{\partial t}+\nabla_x u\Big)(\rho\bar u)+\big(u\cdot\nabla_x+(\nabla_x\cdot u)I\big)(\rho\bar u-\rho u)\\ &=\rho C_{\bar u}+\big[\nabla_x\cdot(\rho\bar u)\big](u-\bar u)+[\nabla_x u-\nabla_x\bar u](\rho\bar u)+(\nabla_x\cdot u)(\rho\bar u-\rho u)+(u\cdot\nabla_x)(\rho\bar u-\rho u)\\ &=\rho C_{\bar u}+\big[(u-\bar u)\cdot\nabla_x\big](\rho\bar u-\rho u)+\big[\nabla_x\cdot(u-\bar u)\big](\rho\bar u-\rho u).\end{aligned}$$

Since $u-\bar u$ is of the order of $\mathrm{O}(\varepsilon)$, we deduce that:

$$\rho C_u+\Big(\frac{\partial}{\partial t}+u\cdot\nabla_x+\nabla_x u+(\nabla_x\cdot u)I\Big)(\rho\bar u-\rho u)=\rho C_{\bar u}+\mathrm{O}\big(\varepsilon^2\big),\tag{73}$$

which implies that (70) is equivalent to (58) up to second order terms in ε. This completes the proof of proposition 4.3. □

Remark 1: Let $f(v) = F(v-u)$, with F given by (47). Then,

$$\int vf\,\mathrm{d}v = \int (v-u)f\,\mathrm{d}v + u\int f\,\mathrm{d}v = \int pF\,\mathrm{d}p + u\int F\,\mathrm{d}p = \rho u + \varepsilon\int pF_1\,\mathrm{d}p + \mathrm{O}(\varepsilon^2).$$

But, with (53), we have

$$\varepsilon\int pF_1\,\mathrm{d}p = -\varepsilon\int_{\mathbb{R}^d}\frac{2\xi}{d}\tau\left(\nabla_x g - C_u\frac{\partial g}{\partial\xi}\right)\mathrm{d}p = \rho(\bar{u}-u).$$

Therefore,

$$\rho\bar{u} = \int vf\,\mathrm{d}v + \mathrm{O}(\varepsilon^2),$$

showing that $\bar{u}$ is the 'true' mean velocity of the distribution function up to terms of order ε^2. This remark explains why the continuity equation (72) and the momentum conservation (second equation of (58)) have a more natural expression once expressed in terms of $\bar{u}$. At variance, the equation for g is simpler in terms of the velocity u about which it is isotropic. The fluid velocity $\bar{u}$ and the 'isotropization velocity' u only differ by order ε terms.

5. System of moments and closure approximations

5.1. Moment system

As mentioned above, the moment system derived from (1), (2) is closed at any order. We shall see that the situation is different for the model (58). In all this section, we assume that $\tau = \tau(t,x,\xi,[g])$ depends on g only through moments of g with respect to the ξ variable. For instance, $\tau(t,x,\xi,[g]) = \tau_1(t,x,\xi,\rho,W)$.

Multiplying the first equation of (58) by $\xi^{k+d/2-1}$ and integrating over ξ we get:

$$\begin{cases} \dfrac{\partial M_k}{\partial t} + u\cdot\nabla_x M_k + \dfrac{2k+d}{d}(\nabla_x\cdot u)M_k = \varepsilon\big[(D_v)_k + (D_h)_k\big], \\ (D_v)_k = \dfrac{4k}{d(d+2)}\left(\left(k+\dfrac{d}{2}\right)M_{\tau,k} + M_{\tau_\xi,k+1}\right)(\sigma(u):\nabla_x u), \\ (D_h)_k = \dfrac{2}{d}\left[\Delta_x M_{\tau,k+1} - \nabla_x\cdot M_{\tau_x,k+1} - \nabla_x\cdot\left(\mathcal{P}\left(\left(k+\dfrac{d}{2}\right)M_{\tau,k} + M_{\tau_\xi,k+1}\right)\right)\right] \\ \qquad - \dfrac{2k}{d}\mathcal{P}\cdot\left[\nabla_x M_{\tau,k} - M_{\tau_x,k} - \mathcal{P}\left(\left(k+\dfrac{d}{2}-1\right)M_{\tau,k-1} + M_{\tau_\xi,k}\right)\right], \end{cases} \tag{74}$$

with the following notations:

$$\tau_x = \nabla_x\tau, \qquad \tau_\xi = \frac{\partial\tau}{\partial\xi}, \qquad \mathcal{P} = \frac{1}{\rho}\nabla_x\left(\frac{2W}{d}\right),$$

and where M_k and $M_{\phi,k}$ are the moments of g defined for an arbitrary function $\phi(\xi)$ by:

$$M_k = \int_{\mathbb{R}^d}\xi^k g(\xi)\,\mathrm{d}p, \qquad M_{\phi,k} = \int_{\mathbb{R}^d}\xi^k\phi(\xi)g(\xi)\,\mathrm{d}p. \tag{75}$$

We recall that $dp = |\mathbb{S}^{d-1}|(2\xi)^{(d-2)/2}\,\mathrm{d}\xi$. For $k = 0$, (74) coincides with the continuity equation (72). Let us compute the order 1 moment equation, we have:

$$\begin{cases} \dfrac{\partial W}{\partial t} + u\cdot\nabla_x W + \dfrac{d+2}{d}(\nabla_x\cdot u)W = \varepsilon\big[(D_v)_1 + (D_h)_1\big], \\ (D_v)_1 = \dfrac{4}{d(d+2)}\left(\dfrac{d+2}{2}M_{\tau,1}, +M_{\tau_\xi,2}\right)\big(\sigma(u):\nabla_x u\big), \\ (D_h)_1 = \dfrac{2}{d}\left[\Delta_x M_{\tau,2} - \nabla_x\cdot M_{\tau_x,2} - \nabla_x\cdot\left(\mathcal{P}\left(\dfrac{d+2}{2}M_{\tau,1} + M_{\tau_\xi,2}\right)\right)\right] \\ \qquad\quad - \dfrac{2}{d}\mathcal{P}\cdot\left[\nabla_x M_{\tau,1} - M_{\tau_x,1} - \mathcal{P}\left(\dfrac{d}{2}M_{\tau,0} + M_{\tau_\xi,1}\right)\right]. \end{cases} \tag{76}$$

Equations (61) and (59) can be expressed in terms of moments:

$$\mu_\tau = \frac{4}{d(d+2)}\left[\frac{d+2}{2}M_{\tau,1} + M_{\tau_\xi,2}\right], \tag{77}$$

and

$$u - \bar{u} = \varepsilon\frac{2}{\rho d}\left[\nabla_x M_{\tau,1} - M_{\tau_x,1} - \mathcal{P}\left(\frac{d}{2}M_{\tau,0} + M_{\tau_\xi,1}\right)\right]. \tag{78}$$

Besides, we can write

$$\frac{\partial W}{\partial t} + \bar{u}\cdot\nabla_x W + \frac{d+2}{d}(\nabla_x\cdot\bar{u})W = \varepsilon\big[\mu_\tau\sigma(\bar{u}):\nabla_x\bar{u} - \nabla_x q\big], \tag{79}$$

where the heat flux vector q is given by:

$$q = -\frac{2}{d}(\nabla_x M_{\tau,2} - M_{\tau_x,2}) + \frac{d+2}{d}\left(\frac{u-\bar{u}}{\varepsilon}W + \frac{\mu_\tau}{\rho}\nabla_x W\right). \tag{80}$$

Collecting the continuity equation (72), the momentum conservation equation (second equation of (58)), and the energy equation (79), we obtain the following extension of the compressible Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \nabla_x\cdot(\rho\bar{u}) = 0, \\ \rho\left(\dfrac{\partial\bar{u}}{\partial t} + (\nabla_x\bar{u})(\bar{u})\right) + \dfrac{2}{d}\nabla_x W = -\varepsilon\nabla_x\cdot\big(\mu_\tau\sigma(\bar{u})\big), \\ \dfrac{\partial W}{\partial t} + \bar{u}\cdot\nabla_x W + \dfrac{d+2}{d}(\nabla_x\cdot\bar{u})W = \varepsilon\big[\mu_\tau\sigma(\bar{u}):\nabla_x\bar{u} - \nabla_x\cdot q\big], \end{cases} \tag{81}$$

where μ_τ and q are respectively the viscosity and the heat flux and are given by (77) and (80). The system (81) is not closed because of the presence of order 2 moments such as $M_{\tau,2}$, $M_{\tau_x,2}$ and $M_{\tau_\xi,2}$, in the expressions of the heat flux q and of the viscosity μ_τ. In the next section we shall investigate some closure hypotheses and compare the obtained system with the usual Navier–Stokes equations. The closure hypotheses are based on the two classes of explicit solutions of the non-dissipative system (1), (2), as discussed in section 2.

5.2. Closure approximations

5.2.1. Dirac closure

We assume that g is the Dirac delta measure (11), that is:

$$g(\xi) = \frac{1}{|\mathbb{S}^{d-1}|}(2\xi)^{1-d/2}\rho(t,x)\delta\big(\xi - e(t,x)\big), \quad e = \frac{W}{\rho}. \tag{82}$$

As explained in section 2, this Ansatz is the most natural bridge between usual gas dynamics and the present multivalued energy model. We have

$$M_{\phi,k} = \int_{\mathbb{R}^d} \xi^k \phi(\xi) g(\xi)\,\mathrm{d}p = \rho e^k \phi(e). \tag{83}$$

Substituting this in (77) and (78), we obtain

$$\mu_\tau = \frac{4}{d(d+2)}\left[\frac{d+2}{2}\rho e\tau(e) + \rho e^2\tau_\xi(e)\right], \qquad \frac{u-\bar{u}}{\varepsilon} = \frac{2}{d}\frac{1}{\rho}\left[\rho e\nabla_x e - \frac{2}{d}e\nabla_x(\rho e)\right]\tau_\xi(e). \tag{84}$$

We also have

$$\nabla_x M_{\tau,2} - M_{\tau_x,2} = \tau(e)\nabla_x(\rho e^2) + \rho e^2\tau_\xi(e)\nabla_x e.$$

Then, using the expression of the heat flux given by (80), we get

$$q = \left[-\frac{2}{d}\rho e\nabla_x(e) + \frac{4}{d^2}e\nabla_x(\rho e)\right]\tau(e) + \left[\frac{4}{d^2}\rho e^2\nabla_x(e) - \frac{8}{d^3}e^2\nabla_x(\rho e)\right]\tau_\xi(e).$$

This can be written in the following factorized form

$$q = \frac{2}{d}e\left[\frac{2}{d}e\tau_\xi(e) - \tau(e)\right]\left[\left(1-\frac{2}{d}\right)\rho\nabla_x e - \frac{2}{d}e\nabla_x\rho\right]. \tag{85}$$

Hence, we recover a 'Fourier–Fick law' for the heat flux:

$$q = -\kappa\nabla_x\tilde{T}, \quad \tilde{T} = e(\rho e)^{-2/d}, \tag{86}$$

$\tilde{T}$ being a generalized temperature, i.e. a quantity such that the heat flux is proportional to the opposite of its gradient. The coefficient

$$\kappa = -\frac{2}{d}(\rho e)^{\frac{d+2}{d}}\left[\frac{2}{d}e\tau_\xi(e) - \tau(e)\right], \tag{87}$$

can be interpreted as a generalized thermal conductivity. A generalized Prandlt number can be defined as the ratio of the viscosity to this thermal conductivity:

$$Pr := \frac{d+2}{2}\frac{\mu_\tau}{\kappa} = \frac{d+2}{2}(\rho e)^{-\frac{2}{d}}\frac{\tau(e) + \frac{2}{d+2}e\tau_\xi(e)}{\tau(e) - \frac{2}{d}e\tau_\xi(e)}. \tag{88}$$

In the case of a power law $\tau(\xi) = \tau_0\xi^{k_0}$, we get:

$$\kappa = \tau_0\frac{2(d-2k_0)}{d^2}\rho^{\frac{d+2}{d}}e^{\frac{d+2}{d}+k_0}, \qquad \mu_\tau = \tau_0\frac{2(d+2+2k_0)}{d(d+2)}\rho e^{k_0+1}, \qquad Pr = \frac{d(d+2+2k_0)}{2(d-2k_0)}\frac{1}{(\rho e)^{2/d}}.$$

We see that the heat conductivity and the viscosity remain positive if and only if $d-2k_0>0$ and $d+2+2k_0>0$, i.e. $-(1+d/2)<k_0<d/2$. In dimension $d=3$, this gives the limitation $-5/2<k_0<3/2$ and the values

$$\kappa=\tau_0\frac{2}{9}(3-2k_0)\rho^{5/3}e^{5/3+k_0},\qquad \mu_\tau=\tau_0\frac{2}{15}(5+2k_0)\rho e^{k_0+1},\qquad Pr=\frac{3}{2}\frac{5+2k_0}{3-2k_0}\frac{1}{(\rho e)^{2/3}}.$$

5.2.2. Maxwellian closure

We investigate the Maxwellian closure which is the natural closure if the kinetic equation (23) corresponds to microscopic molecular motions. It consists in assuming g to be a Maxwellian:

$$g(t,x,\xi)=\frac{\rho(t,x)}{(2\pi T(t,x))^{d/2}}\exp\left(-\frac{\xi}{T(t,x)}\right),\tag{89}$$

and T is related to W by the perfect gas equation-of-state $W=(d/2)\rho T$. We shall restrict the function $\tau(\xi)$ to power laws $\tau(\xi)=\frac{1}{\rho}\tau_0(x,t)\xi^{k_0}$. Then we obviously have $M_{\tau,k}=\frac{1}{\rho}\tau_0 M_{k+k_0}$ and $M_{\tau_\xi,k}=\frac{1}{\rho}\tau_0 k_0 M_{k+k_0-1}$. As g is given by (89) we have $M_k=\alpha_k\rho T^k$, with

$$\alpha_k=\frac{2^{d/2-1}|\mathbb{S}^{d-1}|}{(2\pi)^{d/2}}\Gamma\left(k+\frac{d}{2}\right)=\frac{\Gamma(k+\frac{d}{2})}{\Gamma(\frac{d}{2})},\tag{90}$$

and Γ is the usual Euler function: $\Gamma(x)=\int_0^{+\infty}t^{x-1}\exp(-t)\,\mathrm{d}t$, for $x>0$. In particular $\alpha_0=1$ and $\alpha_{k+1}=(k+(d/2))\alpha_k$, which, when k is an integer, gives $\alpha_k=(d/2)((d/2)+1)\cdots((d/2)+k-1)$. Hence

$$\mu_\tau=\frac{4}{d(d+2)}\left(k_0+\frac{d}{2}+1\right)\tau_0\alpha_{k_0+1}T^{k_0+1},\tag{91}$$

and

$$\frac{u-\bar u}{\varepsilon}=\frac{2\tau_0}{\rho^2 d}\left[\alpha_{k_0+1}\nabla_x(\rho T^{k_0+1})-\alpha_{k_0}\left(\frac{d}{2}+k_0\right)T^{k_0}\nabla_x(\rho T)\right].\tag{92}$$

Now we insert these formulas into the expression of the heat flux given by (80) and, after some computation we obtain

$$q=-\frac{\tau_0}{d}(2k_0^2+2k_0+d+2)\alpha_{k_0+1}T^{k_0+1}\nabla_x T,\tag{93}$$

which has the form of a 'Fourier–Fick law': $q=-\kappa\nabla_x T$, where T is now the usual temperature, and where

$$\kappa=\frac{\tau_0}{d}(2k_0^2+2k_0+d+2)\alpha_{k_0+1}T^{k_0+1}.\tag{94}$$

Now using the expression (91) we get the following value for the Prandlt number:

$$Pr:=\frac{d+2}{2}\frac{\mu_\tau}{\kappa}=\frac{2k_0+d+2}{2k_0+d+2+2k_0^2}.\tag{95}$$

Hence, for $k_0\in[0,+\infty]$, the Prandlt number can take all the values of the interval $]0,1]$. In particular the value $Pr=2/3$, which is the physically realistic value for monoatomic gases for $d=3$, is reached for $k_0=\frac{1}{4}(1+\sqrt{21})\approx 1.4$.

5.3. Closure approximations for higher order moment system

We consider the moment system (74) truncated to the order $N \geqslant 2$, i.e. the system of equations (74) of unknowns M_k, for $0 \leqslant k \leqslant N$. We know that this system of $N+1$ equations is not a closed system because of the presence of moments of the kind $M_{\phi,k}$ with $0 \leqslant k \leqslant N$, and of the kind $M_{\phi,N+1}$. In this section we show how to derive an approximate closed system of $N+1$ equations. For that purpose, we assume that the function g is not far from either a Dirac or a Maxwellian distribution.

First we analyse the Dirac closure. We have to express the moments $M_{\phi,k}$, $0 \leqslant k \leqslant N+1$, in terms of moments M_k for $0 \leqslant k \leqslant N$. It is clear from (83) that:

$$M_{\phi,k} = \phi(e) M_k, \quad \text{for } 0 \leqslant k \leqslant N; \qquad M_{\phi,N+1} = e\phi(e) M_N, \tag{96}$$

where we recall that $e = W/\rho$ and ϕ is an arbitrary function of ξ. Substituting these two relations in (74) for $0 \leqslant k \leqslant N$, we obtain a closed system of unknowns $(M_k)_{0\leqslant k\leqslant N}$.

Now consider a Maxwellian closure approximation and suppose a power law relation $\tau(\xi) = \tau_0 \xi^{k_0}$ with $k_0 \in \mathbb{N}$. In this case:

$$M_{\tau,k} = \tau_0 M_{k+k_0}, \qquad M_{\tau_x,k} = (\tau_0)_x M_{k+k_0}, \qquad M_{\tau_\xi,k} = \tau_0 k_0 M_{k+k_0-1}.$$

Thus, to obtain a closed system of moments $(M_k)_{0\leqslant k\leqslant N}$, we have to express the moments M_{k+k_0} or M_{k+k_0+1} for $0 \leqslant k \leqslant N$ in terms of moments $(M_k)_{0\leqslant k\leqslant N}$ only. We use that $M_k = \alpha_k \rho T^k$, and deduce the relation:

$$M_{k+s} = \frac{\alpha_{k+s}}{\alpha_k} T^s M_k, \tag{97}$$

for $0 \leqslant k \leqslant N$, $s \in \mathbb{N}$. This relation enables us to substitute moments M_k, $k_0 \leqslant k \leqslant k_0 + N + 1$, appearing in (74), by moments M_k for $0 \leqslant k \leqslant N$. We then obtain a closed system of unknowns $(M_k)_{0\leqslant k\leqslant N}$.

Whether these moment systems give rise to well-posed problems is of course a very important question, which we shall defer to future work.

6. Extension to more general wave-particle collision operators

6.1. A collision operator with angular dependence

We consider the following extension of the collision operator (14), (15):

$$Q(f,u) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} \sigma\left(\omega, \omega', \frac{|v-u|^2}{2}, [f]\right) \left[f\left(u + |v-u|\omega'\right) - f\left(u + |v-u|\omega\right)\right] \mathrm{d}\omega'. \tag{98}$$

We shall assume that the cross section σ is rotationally invariant, i.e. has the form: $\sigma(\omega, \omega') = \sigma_0(\omega \cdot \omega') > 0$. As for the function τ in (14), σ_0 is an operator acting on the velocity dependence of f such that $\sigma_0(x, t, \omega \cdot \omega', [f])$ depends on v only through $|v-u|^2/2$. Again, an example is $\sigma_0(\omega \cdot \omega', [f]) = \sigma_0'(\omega \cdot \omega', P_u(f))$ where σ_0' is a local function of $P_u(f)$. More complex examples can also be envisaged. In this section we shall see how macroscopic models associated with this collision operator differ from those obtained so far. First we state the following:

PROPOSITION 6.1: (i) *We have*

$$\int_{\mathbb{S}^{d-1}} \sigma_0(t,x,\omega\cdot\omega',\xi,[f])(\omega'-\omega)\,\mathrm{d}\omega' = -(\tau_1)^{-1}(t,x,\xi,[f])\omega, \tag{99}$$

where

$$(\tau_1)^{-1}(t,x,\xi,[f]) = \int_{\mathbb{S}^{d-1}} (1-\omega'\cdot\omega)\sigma_0(t,x,\omega\cdot\omega',\xi,[f])\,\mathrm{d}\omega', \tag{100}$$

is independent of ω, and is the so-called momentum transfer frequency.

(ii) *The collision operator* (98) *satisfies the following conservation properties*:
(a)

$$\int_{\mathbb{S}^{d-1}} Q(f,u)(u+|v-u|\omega)\,\mathrm{d}\omega = 0,$$

(b) *$Q(f,u)$ is momentum-preserving if and only if u and f are linked by the relation*:

$$\int_{\mathbb{R}^d} (v-u)(\tau_1)^{-1}(t,x,\xi,[f])f(v)\,\mathrm{d}v = 0, \tag{101}$$

where τ_1 is defined by (100).

The proof is standard and is omitted.

Now, let us perform the change to the local fluid frame (29). The expression of the collision operator is changed to

$$LF(p) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} \sigma_0(\omega\cdot\omega',\xi,F)\big[F(|p|\omega') - F(|p|\omega)\big]\,\mathrm{d}\omega', \tag{102}$$

which is a linear operator if the dependence of the cross section upon f is freezed. As $\sigma_0 > 0$, its null-space is clearly the space of isotropic functions. This is easily shown from the following weak formulation:

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} LF(|p|\omega)\phi(|p|\omega)\,\mathrm{d}\omega\\ &\quad= -\frac{1}{2}\frac{1}{|\mathbb{S}^{d-1}|}\int\int_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} \sigma_0(\omega\cdot\omega',\xi,F)\big[F(|p|\omega') - F(|p|\omega)\big]\times\big[\phi(|p|\omega') - \phi(|p|\omega)\big]\,\mathrm{d}\omega\,\mathrm{d}\omega',\end{aligned} \tag{103}$$

for all test functions ϕ.

Following the method used in section 2.3, we can write similarly to (52):

$$\begin{cases}\Pi\mathcal{A}(g,u) + \varepsilon\Pi\mathcal{A}\big(L^{-1}\big[(I-\Pi)\mathcal{A}(g,u)\big],u\big) = 0,\\ \mathcal{B}(g,u) + \varepsilon\mathcal{B}\big(L^{-1}\big[(I-\Pi)\mathcal{A}(g,u)\big],u\big) = 0.\end{cases} \tag{104}$$

Using the expression of $(I-\Pi)\mathcal{A}(g,u)$ given by (62), we obtain

$$\begin{cases}\Pi\mathcal{A}(g,u) + \varepsilon\Pi\mathcal{A}\left(\left(\nabla_x g - C_u\dfrac{\partial g}{\partial\xi}\right)h(p) - \dfrac{\partial g}{\partial\xi}\mathcal{U} : H(p), u\right) = 0,\\ \mathcal{B}(g,u) + \varepsilon\mathcal{B}\left(\left(\nabla_x g - C_u\dfrac{\partial g}{\partial\xi}\right)\cdot h(p) - \dfrac{\partial g}{\partial\xi}\mathcal{U} : H(p)\,,\, u\right) = 0,\end{cases}$$

where $h(p)$ and $H(p)$ are respectively vector and matrix functions, solutions to:

$$Lh(p) = p \quad \text{and} \quad LH(p) = p \otimes p - \frac{2\xi}{d} I. \tag{105}$$

The resolution of (105) with L given by (102) gives:

$$h(p) = -\tau_1 p \quad \text{and} \quad H(p) = -\tau_2 \left[p \otimes p - \frac{2\xi}{d} I \right], \tag{106}$$

where τ_1 is given by (100) with $f = g$ and τ_2 is given by:

$$(\tau_2)^{-1}(t, x, \xi, [g]) = \frac{d-1}{d} \int_{\mathbb{S}^{d-1}} \sigma_0\big(t, x, \omega \cdot \omega', \xi, [g]\big)\big(1 - (\omega \cdot \omega')^2\big)\, \mathrm{d}\omega'. \tag{107}$$

Then the computations lead to the following macroscopic model:

$$\begin{cases} \dfrac{\partial g}{\partial t} + u \cdot \nabla_x g - (\nabla_x \cdot u)\dfrac{2\xi}{d}\dfrac{\partial g}{\partial \xi} \\ \quad = \varepsilon \dfrac{2}{d} \xi^{1-d/2} \left(\nabla_x - C_u \dfrac{\partial}{\partial \xi} \right) \left(\tau_1(\xi, g) \xi^{d/2} \left(\nabla_x - C_u \dfrac{\partial}{\partial \xi} \right) g \right) \\ \quad + \varepsilon \dfrac{4}{d(d+2)} \xi^{1-d/2} \dfrac{\partial}{\partial \xi} \left(\tau_2(\xi, g) \xi^{\frac{d+2}{2}} \dfrac{\partial g}{\partial \xi} g \right) \big(\sigma(u) : (\nabla_x u)\big), \\ \rho \left(\dfrac{\partial \bar{u}}{\partial t} + (\nabla_x \bar{u})(\bar{u}) \right) + \dfrac{2}{d} \nabla_x W = -\varepsilon \nabla_x \cdot \big(\mu_{\tau_2} \sigma(\bar{u})\big), \end{cases} \tag{108}$$

with

$$\rho(\bar{u} - u) = -\varepsilon \int_{\mathbb{R}^d} \frac{2\xi}{d} \tau_1 \left(\nabla_x g - C_u \frac{\partial g}{\partial \xi} \right) \mathrm{d}p, \tag{109}$$

and μ_{τ_2} given by (61) with τ replaced by τ_2. We notice that the model is of the same form as (58), but with two different characteristic times: the first one τ_1 is for the heat dissipation operator and the other one τ_2, for the viscosity operator.

6.2. A Fokker–Planck type operator

An interesting limiting case of the angular dependent collision operators (98) is when the cross section σ concentrates on small angle deviations, i.e. behaves like a Dirac delta measure of the set $\omega = \omega'$. In this so-called 'grazing collision limit' (see, e.g., [16] and more recently, [17,18] for the case of the Landau limit of the classical Boltzmann equation), the operator (98) converges to a diffusion operator in the angle variable. Indeed, let the cross section $\sigma_\eta(\omega, \omega') = \bar{\sigma}_0(|\omega - \omega'|/\eta)$ be parametrized by the parameter $\eta > 0$ and let Q_η be the corresponding collision operator according to (98). Then, formally, at the leading order when η goes to 0, Q_η is proportional to the following Fokker–Planck type collision operator:

$$\begin{cases} Q_0(f, u) = \nabla_v \cdot \big[\Phi(v - u) \nabla_v f \big], \\ \Phi(w) = (d-1)(\tau_1)^{-1} \left(t, x, \dfrac{|w|^2}{2}, f \right) \big(|w|^2 I - w \otimes w \big), \end{cases} \tag{110}$$

with τ_1 given by (100). To some extent, the cases of the angle-independent collision time operator (14), (15) and of the grazing collision limit (110) are two opposite limiting cases

The integral of Q_0 against all functions of $\xi = |v-u|^2/2$ vanishes because $\ker \Phi(w) = \mathbb{R}w$. The Fokker–Planck operator is momentum-preserving if and only if f and u are linked by (101). With the change to the local frame (29), the collision operator is changed into

$$LF = \nabla_p \cdot \left[\Phi(p)\nabla_p F\right]. \tag{111}$$

The computations are the same as in section 6.1 and we find the macroscopic model (108) in which the two time scales τ_1 and τ_2 are now linked by

$$\tau_2 = \frac{d-1}{2d}\tau_1 .$$

7. Conclusion

In this work, we have discussed the properties of a gas dynamics model in which the energy in the frame moving with the fluid is multivalued (or more precisely, measure-valued). This model is derived from a kinetic model involving a collision operator which describes the isotropization of the distribution function about the fluid velocity. We have been specifically concerned with the establishment of the dissipative corrections to the purely convective multivalued energy model, in the case where the isotropization rate is a function of the particle energy. In this case, the moment system associated with the so-obtained model appears close to the usual compressible Navier–Stokes equations but with expressions of the viscosity and heat flux depending on the details of the energy distribution of the particles. Two closures of the moment equations have been investigated: the Dirac and the Maxwellian closures. In both cases, the expressions of the viscosity and thermal conduction are different from the usual ones. We have tried to give arguments why these features could be useful in an attempt to model fluid turbulence from kinetic theory. The future directions of this work are twofold: first we shall try to validate these ideas against numerical solutions of well established turbulent flows; second, we shall attempt to improve the physical relevance of the model by better including the already known phenomenology of turbulence.

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